

# BIRATIONAL GEOMETRY OF SINGULAR FANO HYPERSURFACES

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**ABSTRACT.** We establish birational superrigidity for a large class of singular projective Fano hypersurfaces of index one. In the special case of isolated singularities, our result applies for instance to: (1) hypersurfaces with semi-homogeneous singularities of multiplicity roughly bounded by half of the dimension of the hypersurface, (2) hypersurfaces with isolated singularities whose Tyurina numbers satisfy a similar bound, and (3) hypersurfaces with isolated singularities whose dual variety is a hypersurface of degree close enough to the expected degree.

## 1. INTRODUCTION

Completing a series of works which began with Iskovskikh and Manin’s theorem on smooth quartic threefolds [IM71] and continued throughout the years in the papers [Puk87, Puk98, Che00, Puk02, dFEM03], it was recently proven in [dF12] that all smooth hypersurfaces  $V$  of degree  $N$  in  $\mathbb{P}^N$ , for  $N \geq 4$ , are birationally superrigid. This means that there are no birational modifications of  $V$  into Mori fiber spaces other than isomorphisms, and implies that  $V$  is not rational. Since no other smooth Fano hypersurface is birationally superrigid, one deduces from this result the complete list of smooth hypersurfaces with such property.

In the present paper we want address the problem for singular hypersurfaces, a setting that is still far from being well understood.

In low dimensions, there are results on quartic threefolds and sextic fivefolds with mild singularities (mostly ordinary double points) obtained in [Puk88, CM04, Mel04, Che07]. A contribution in higher dimensions was given by Pukhlikov in [Puk02b, Puk03], where hypersurfaces with *semi-homogeneous singularities*<sup>1</sup> are studied under a certain “regularity” condition requiring that, at each point of the variety, the intermediate homogeneous terms of the local equation of the hypersurface form a regular sequence.

Our goal is to extend the methods introduced in [dF12] to study the problem without the use of special conditions on the local equations of  $X$  and to allow for more general classes of singularities, including positive dimensional ones. The following defines the type of condition on singularities we consider.

**Definition 1.1.** Let  $P \in V$  be a germ of a normal variety. For any pair of integers  $(\delta, \nu)$  with  $\delta \geq -1$  and  $\nu \geq 1$ , we say that  $P$  is a *singularity of type*  $(\delta, \nu)$  if the singular locus has dimension at most  $\delta$  and, given a general complete intersection  $X \subset V$  of codimension  $\delta + 2$  through  $P$ , the  $(\nu - 1)$ -th power of the maximal ideal  $\mathfrak{m}_{X,P} \subset \mathcal{O}_X$  is contained in the integral closure of the Jacobian ideal  $\text{Jac}_X := \text{Fitt}^{\dim X}(\Omega_X^1) \subset \mathcal{O}_X$  of  $X$ .

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<sup>1</sup>Also known as *ordinary multiple points*, these are isolated hypersurface singularities whose tangent cones are smooth away from the vertex.

For instance, regular points are singularities of type  $(-1, 1)$  and semi-homogeneous hypersurface singularities of multiplicity  $\nu$  are of type  $(0, \nu)$ . More generally, every isolated hypersurface singularity of multiplicity  $\nu$  whose tangent cone is smooth away from a set of dimension two is a singularity of type  $(0, \nu)$ . In general, singularities of type  $(\delta, \nu)$  are also of type  $(\delta', \nu')$  for every  $\delta' \geq \delta$  and  $\nu' \geq \nu$ .

We can now state our main result.

**Theorem 1.2.** *Let  $V \subset \mathbb{P}^N$  be a hypersurface of degree  $N$  with only singularities of type  $(\delta, \nu)$ , and assume that*

$$\delta + \nu \leq \frac{1}{2}N - 3.$$

*Then  $V$  is a birationally superrigid Fano variety with Picard number one and factorial terminal singularities. In particular,  $V$  is not rational and  $\text{Bir}(V) = \text{Aut}(V)$ .*

To illustrate this result, we present a few special cases where the singularities are isolated. We start with the case of semi-homogeneous singularities.

**Corollary 1.3.** *Every hypersurface  $V \subset \mathbb{P}^N$  of degree  $N$  with semi-homogeneous singularities of multiplicity at most  $\frac{1}{2}N - 3$  is birationally superrigid.*

Comparing this with the results of Pukhlikov, one sees that while the bounds on multiplicity in the corollary are more restrictive than those in his papers, no “regularity” assumption is required in our result. Furthermore, the hypothesis on the singularities being semi-homogeneous can be relaxed by allowing, for instance, the tangent cones to have singularities in dimension one or two.

Another special case of the theorem can be formulated in terms of the Tyurina numbers of the singularities. For every integer  $i$  and every point  $P \in V$ , denote by  $\tau_P^{(i)}(V)$  the Tyurina number (at  $P$ ) of a general complete intersection of codimension  $i$  in  $V$  passing through  $P$ .

**Corollary 1.4.** *Let  $V \subset \mathbb{P}^N$  be a hypersurface of degree  $N$  with isolated singularities, and assume that for every  $P \in V$*

$$\min\{\tau_P(V), \tau'_P(V), \tau''_P(V)\} \leq \frac{1}{2}N - 4.$$

*Then  $V$  is birationally superrigid.*

Since the Tyurina number is bounded above by the Milnor number, a similar corollary can be formulated in terms of the Milnor numbers of general restrictions of  $V$ , which are known as the *Teissier–Milnor numbers* of  $V$ . Using then a result of Teissier, we obtain the following interesting consequence.

**Corollary 1.5.** *Let  $V \subset \mathbb{P}^N$  be a hypersurface of degree  $N$  with isolated singularities, and assume that the dual variety  $\check{V} \subset \mathbb{P}^N$  (defined as the closure in  $\mathbb{P}^N$  of the set of points representing the tangent hyperplanes to  $V$  at regular points) is a hypersurface of degree*

$$\deg \check{V} \geq N(N-1)^{N-1} - (N+2s-10)$$

*where  $s$  is the number of singular points. Then  $V$  is birationally superrigid.*

The interest in birational rigidity originates from the realization that, differently from the surface case, higher dimensional Fano varieties and Mori fiber spaces present a wide spectrum of possible birational characteristics, with rational varieties at one end of the spectrum and birationally superrigid varieties at the other end. The problem of determining birational links between different Mori fiber spaces, or the lack thereof, finds its motivation in the Minimal

Model Program and can be viewed as the counterpart of the question asking about the unicity of minimal models.

Birational rigidity has been extensively studied in dimension three, and several examples of birationally rigid Fano manifolds are also known in higher dimensions. This property is however sensitive to the singularities. For instance, smooth quartic threefolds are birationally superrigid, but those with a double point are only birationally rigid as the projection from the point induces a birational automorphism. Similarly, quartic threefolds that are singular (with multiplicity 3) along a line can be birationally modified into conic bundles.

Fano hypersurfaces provide a good setting where to explore the problem in the presence of singularities. The aforementioned works on quartic threefolds show that in low dimensions the problem becomes rather delicate already when dealing with mild singularities. The theorem we prove in this paper should be viewed as complementing those studies by showing that the problem stabilizes in the best possible way when the dimension is let grow and the “depth” of the singularities is maintained, in some sense, asymptotically bounded in terms of the dimension.

One advantage of our methods is that they involve, at a certain point of the proof, the projection of a linear section of the hypersurface onto a projective space. We use the projection to get rid of the singularities of the hypersurface, and the valuative contribution of the Jacobian ideal is the only piece of information we need to keep track of. It is thanks to this that we can allow such a large variety of singularities in the theorem.

Properties of singularities of type  $(\delta, \nu)$  and the corollaries are discussed in Section 2. The proof of the theorem is then addressed in the remaining three sections. Each of these sections starts with a brief overview (written in *italic*) of the basic terminology and of some of the key results needed in the corresponding portion of the proof. For other properties used in the proof we provide the appropriate reference within the proof itself.

All varieties are assumed to be defined over the complex numbers.

## 2. SINGULARITIES OF TYPE $(\delta, \nu)$ .

In this section we discuss some properties of singularities of type  $(\delta, \nu)$  introduced in Definition 1.1 and give the proofs of the three corollaries stated in the introduction.

Given a germ of an isolated singularity  $P \in X$ , we define

$$\nu_P(X) := \min \{ \nu \in \mathbb{N} \mid (\mathfrak{m}_{X,P})^{\nu-1} \subset \overline{\text{Jac}_X} \},$$

where the bar in the right-hand side denotes integral closure. Note that a normal singularity  $P \in V$  is of type  $(\delta, \nu)$  if and only if the singular locus has dimension at most  $\delta$  and  $\nu_P(X) \leq \nu$  for a general complete intersection  $X \subset V$  of codimension  $\delta + 2$  through  $P$ .

**Proposition 2.1.** *Let  $P \in X$  be an isolated singularity. Then for every general hyperplane section  $H \subset X$  through  $P$  we have*

$$\nu_P(H) \leq \nu_P(X).$$

*Proof.* It follows by Teissier’s Idealistic Bertini Theorem [Tei77, 2.15. Corollary 3], which implies that

$$\overline{\text{Jac}_X} \cdot \mathcal{O}_H \subset \overline{\text{Jac}_X \cdot \mathcal{O}_H} = \overline{\text{Jac}_H},$$

and the fact that  $\mathfrak{m}_{X,P} \cdot \mathcal{O}_H = \mathfrak{m}_{H,P}$ . □

**Corollary 2.2.** *A singularity of type  $(\delta, \nu)$  is also of type  $(\delta', \nu')$  for every  $\delta' \geq \delta$  and  $\nu' \geq \nu$ .*

Our focus in this paper is on hypersurface singularities. Let us thus assume that  $P \in X$  is an isolated hypersurface singularity. For ease of notation, we consider the case where  $X$  is a hypersurface in an affine space  $\mathbb{A}^n$  with an isolated singularity at  $P$ . We fix affine coordinates  $(x_1, \dots, x_n)$  centered at  $P$ , and let  $h(x_1, \dots, x_n) = 0$  be an equation defining  $X$ . Then the Jacobian ideal of  $X$  is cut out, on  $X$ , by the partial derivatives of  $h$ :

$$\text{Jac}_X = \left( \frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n} \right) \cdot \mathcal{O}_X.$$

A special case where  $\nu_P(X)$  is easy to compute is when  $P \in X$  is a semi-homogeneous hypersurface singularity. Recall that the *multiplicity*  $e_P(X)$  of  $X$  at  $P$  is the degree of the tangent cone  $C_P X$ .

**Proposition 2.3.** *If  $P \in X$  is a semi-homogeneous hypersurface singularity, then*

$$\nu_P(X) = e_P(X).$$

*Proof.* Let for short  $m := e_P(X)$ . Let  $f: \tilde{X} \rightarrow X$  and  $g: \tilde{\mathbb{A}}^n \rightarrow \mathbb{A}^n$  be the blow-ups of  $P$ , and let  $F$  and  $G$  be the respective exceptional divisors. Then  $\tilde{X} \subset \tilde{\mathbb{A}}^n$  is the proper transform of  $X$  and  $g^*X = \tilde{X} + mG$ . In particular,  $\text{mult}_P(h) = m$ , and thus  $\text{mult}_P(\partial h / \partial x_i) = m - 1$ . By hypothesis,  $F = \tilde{X} \cap G$  is a smooth hypersurface of degree  $m$  in  $G \cong \mathbb{P}^{n-1}$ , defined by the vanishing of the degree  $m$  homogeneous form  $h_m$  of  $h$ . It follows that the homogeneous ideal

$$\left( \frac{\partial h_m}{\partial x_1}, \dots, \frac{\partial h_m}{\partial x_n} \right) \subset \mathbb{C}[x_1, \dots, x_n]$$

has no zeroes in  $\mathbb{P}^{n-1}$ . This implies that  $\text{Jac}_X \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-(m-1)F)$ , and thus  $\overline{\text{Jac}_X} = f_* \mathcal{O}_{\tilde{X}}(-(m-1)F)$ . The assertion follows then by the fact that  $(\mathfrak{m}_{X,P})^k \cdot \mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-kE)$ .  $\square$

*Proof of Corollary 1.3.* By Proposition 2.3,  $P \in V$  is a singularity of type  $(0, e_P(X))$  for a general complete intersection  $X \subset V$  of codimension two passing through  $P$ . Since  $e_P(X) = e_P(V)$ , the corollary follows directly from Theorem 1.2.  $\square$

The Jacobian ideal retains important information of a singularity. For instance, it is a theorem of Mather and Yau [MY82] that, for an isolated hypersurface singularity  $P \in X$ , the Jacobian  $\mathbb{C}$ -algebra  $\mathcal{O}_{X,P} / \text{Jac}_X$  determines the analytic isomorphism class of the singularity. The dimension of this algebra is called the *Tyurina number* of the singularity. If, as above,  $X$  is defined by  $h(x_1, \dots, x_n) = 0$  in  $\mathbb{A}^n$  and  $P = (0, \dots, 0)$ , then the Tyurina number is given by

$$\tau_P(X) := \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1, \dots, x_n]]}{\left( h, \frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n} \right)}.$$

The Tyurina number is closely related to the *Milnor number* of the singularity, which is the number of spheres in the bouquet homotopically equivalent to the Milnor fiber and is computed by the dimension

$$\mu_P(X) := \dim_{\mathbb{C}} \frac{\mathbb{C}[[x_1, \dots, x_n]]}{\left( \frac{\partial h}{\partial x_1}, \dots, \frac{\partial h}{\partial x_n} \right)}.$$

For every  $i$ , we define the  $i$ -th *Tyurina number*  $\tau_P^{(i)}(X)$  and the  $i$ -th *Teissier–Milnor number*  $\mu_P^{(i)}(X)$  of  $X$  at  $P$  to be, respectively, the Tyurina number and the Milnor number of a general complete intersection of codimension  $i$  passing through  $P$  (there is a difference here with respect to the notation originally used by Teissier where the index  $i$  refers to dimension rather than codimension).

**Proposition 2.4.** *With the above notation, we have*

$$\nu_P(X) \leq \tau_P(X) + 1$$

*Proof.* Let for short  $\nu := \nu_P(X)$ . By definition, we have  $(\mathfrak{m}_{X,P})^{\nu-1} \not\subset \overline{\text{Jac}_X}$ . In view of the valuative interpretation of integral closure, this means that there is a valuation  $v$  on the function field of  $X$  (which one may assume to be divisorial) with center  $P$  such that

$$(\nu - 1) \cdot v(\mathfrak{m}_{X,P}) < v(\text{Jac}_X).$$

Consider the sequence of ideals  $\mathfrak{q}_k := \mathfrak{m}^k + \text{Jac}_X \subset \mathcal{O}_X$ . Since  $v(\mathfrak{q}_k) = k \cdot v(\mathfrak{m}_{X,P})$  for  $1 \leq k \leq \nu - 1$ , we have a chain of strict inclusions of ideals

$$\mathcal{O}_X \supsetneq \mathfrak{q}_1 \supsetneq \mathfrak{q}_2 \supsetneq \cdots \supsetneq \mathfrak{q}_{\nu-1} \supsetneq \text{Jac}_X.$$

This implies that  $\tau_P(X) \geq \nu - 1$ .  $\square$

*Proof of Corollary 1.4.* Let  $P \in V$  be one of the singularities of  $V$ , and let  $i \in \{0, 1, 2\}$  be such that  $\tau_P^{(i)}(V) \leq \frac{1}{2}N - 4$ . If  $V^{(i)} \subset V$  denotes a general complete intersection of codimension  $i$  through  $P$ , then we have  $\nu_P(V^{(i)}) \leq \frac{1}{2}N - 3$  by Proposition 2.4. Since  $i \leq 2$ , this implies by Proposition 2.1 that if  $X = V'' \subset V$  is a general complete intersection of codimension two then  $\nu_P(X) \leq \frac{1}{2}N - 3$ . Therefore the corollary follows from Theorem 1.2.  $\square$

*Proof of Corollary 1.5.* Let  $P_1, \dots, P_s \in V$  be the singular points. It is proven in [Tei80, Appendix 2] that the dual variety has degree

$$\deg \check{V} = N(N-1)^{N-1} - \sum_{j=1}^s (\mu_{P_j}(V) + \mu'_{P_j}(V)).$$

Note that, for every  $j$ , both  $\mu_{P_j}(V)$  and  $\mu'_{P_j}(V)$  are positive integers. Then, for any given  $j$  we have

$$\mu_{P_j}(V) + \mu'_{P_j}(V) \leq N(N-1)^{N-1} - 2(s-1) - \deg \check{V} \leq N-8,$$

and hence

$$\min\{\mu_{P_j}(V), \mu'_{P_j}(V)\} \leq \frac{1}{2}N - 4.$$

Since  $\tau_{P_j}^{(i)}(V) \leq \mu_{P_j}^{(i)}(V)$ , one can then apply Corollary 1.4.  $\square$

### 3. SETTING UP THE PROOF OF THEOREM 1.2

**3.1. Background: Singularities of pairs.** A standard reference is [Kol97]. Let  $X$  be a normal variety with  $\mathbb{Q}$ -Cartier canonical class, and let  $E$  be a prime divisor on a resolution  $f: \tilde{X} \rightarrow X$ . We say that  $E$  is a divisor over  $X$ . The image of  $E$  in  $X$  is the center of  $E$ ; the divisor is exceptional over  $X$  if the center has codimension  $\geq 2$ . The divisor defines a valuation  $\text{val}_E$  over  $X$ ; if  $Z \subset X$  is a proper closed subscheme and  $I_Z \subset \mathcal{O}_X$  is its ideal sheaf, then we set  $\text{val}_E(Z) := \text{val}_E(I_Z)$ . The discrepancy of  $X$  along  $E$  is the coefficient  $k_E(X) := \text{ord}_E(K_{\tilde{X}/X})$  of  $E$  in the relative canonical divisor  $K_{\tilde{X}/X}$ . Given a finite, formal  $\mathbb{R}$ -linear combination  $Z = \sum c_i Z_i$  of proper closed subschemes  $Z_i \subset X$ , the log discrepancy of the pair  $(X, Z)$  along  $E$  is defined to be

$$a_E(X, Z) := k_E(X) + 1 - \sum c_i \text{val}_E(Z_i).$$

The minimal log discrepancy of  $X$  at a proper closed subset  $T \subset X$  is the infimum of all log discrepancies along divisors with center in  $T$ , and is denoted by  $\text{mld}(T; X, Z)$ .

The pair  $(X, Z)$  is log canonical (resp., log terminal) if  $a_E(X, Z) \geq 0$  (resp.,  $a_E(X, Z) > 0$ ) for all  $E$  over  $X$ . The pair is canonical (resp., terminal) if  $a_E(X, Z) \geq 1$  (resp.,  $a_E(X, Z) > 1$ ) for all  $E$  exceptional over  $X$ . Assuming that  $X$  has canonical singularities and  $c_i > 0$  for all  $i$ , we define the canonical threshold  $\text{can}(X, Z)$  of  $(X, Z)$  to be the largest number  $c$  such that  $(X, cZ)$  is canonical.

**Theorem 3.1** (Inversion of Adjunction [EM04]). *Suppose that  $X$  is a locally complete intersection variety, and let  $Y \subset X$  be a normal effective Cartier divisor that is not contained in  $\bigcup Z_i$ . Assume that  $c_i > 0$  for all  $i$ . Then for every proper closed subset  $T \subset Y$  we have*

$$\text{mld}(T; X, Z + Y) = \text{mld}(T; Y, Z|_Y).$$

A Mori fiber space is a normal projective variety  $X$  with  $\mathbb{Q}$ -factorial terminal singularities, equipped with an external Mori contraction of fiber type  $X \rightarrow S$  (so that  $\dim S < \dim X$ ,  $\text{rk Pic}(S) = \text{rk Pic}(X) - 1$ , and  $-K_X$  is relatively ample over  $S$ ). A Mori fiber space is said to be birationally superrigid if there are no birational maps to other Mori fiber spaces other than isomorphisms.

**Theorem 3.2** (Noether–Fano Inequality [IM71, Cor95]). *Assume that  $X$  is a Fano variety of Picard number one and terminal  $\mathbb{Q}$ -factorial singularities, and suppose that there is a birational map  $\phi: X \dashrightarrow X'$  where  $X'$  is a Mori fiber space. Fix an embedding  $X' \subset \mathbb{P}^m$ , let  $\mathcal{H} := \phi_*^{-1}|\mathcal{O}_{X'}(1)|$  be the linear system on  $X$  giving the map  $X \dashrightarrow X' \hookrightarrow \mathbb{P}^m$ , and let  $B(\mathcal{H}) \subset X$  be its base scheme. Let  $r$  be the rational number such that  $\mathcal{H} \subset |-rK_X|$ . Then*

$$\text{can}(X, B(\mathcal{H})) < 1/r.$$

**3.2. Step 1 of the proof: Cutting down the singular locus.** Let  $V \subset \mathbb{P}^N$  be as in Theorem 1.2.

**Lemma 3.3.**  *$V$  is a normal, factorial Fano variety of index one and Picard number one.*

*Proof.* First note that, by definition of singularity of type  $(\delta, \nu)$ ,  $V$  is normal. By the Lefschetz Hyperplane Theorem,  $\text{Pic}(V)$  is generated by the class of  $\mathcal{O}_V(1)$ , thus  $V$  has Picard number one and, by adjunction, is a Fano variety of index one. Consider a general linear 4-space  $\mathbb{P}^4 \subset \mathbb{P}^N$ , and let  $W \subset \mathbb{P}^4$  be the hypersurface cut out by  $V$ . Note that  $W$  is smooth. Applying then the Lefschetz Hyperplane Theorem to  $W \subset \mathbb{P}^4$  we see that  $\text{Pic}(W)$  is also generated by the hyperplane class, and hence the restriction map  $\text{Pic}(V) \rightarrow \text{Pic}(W)$  is an isomorphism. Since  $W$  is smooth, the class map  $\text{Pic}(W) \rightarrow \text{Cl}(W)$  is an isomorphism. On the other hand, the restriction of Weil divisors (which is well-defined in our setting) induces an isomorphism  $\text{Cl}(V) \rightarrow \text{Cl}(W)$  by an inductive application of [RS06, Theorem 1]. It follows that  $\text{Pic}(V) \rightarrow \text{Cl}(V)$  is an isomorphism.  $\square$

We start by assuming that  $V$  has terminal singularities. We will argue at the end of this section that essentially the same proof given under this condition also proves that  $V$  must have terminal singularities in the first place.

Under the assumption that  $V$  has terminal singularities, it follows by Lemma 3.3 that  $V$  is a Mori fiber space (over a point), and we can thus inquire about whether it is birationally superrigid. We shall assume that this is not the case, and proceed by contradiction. Then there is a birational map

$$\phi: V \dashrightarrow V'$$

from  $V$  to a Mori fiber space  $V'$  that is not an isomorphism. Fix a projective embedding  $V' \subset \mathbb{P}^m$ , and let  $\mathcal{H} = \phi_*^{-1}|\mathcal{O}_{V'}(1)|$ . Note that  $\mathcal{H} \subset |\mathcal{O}_V(r)|$  for some integer  $r \geq 1$ . Let

$B(\mathcal{H})$  be the base scheme of  $\mathcal{H}$ , and let

$$D \in \mathcal{H} \quad \text{and} \quad B = D_1 \cap D_2 \subset V$$

be, respectively, a general member of  $\mathcal{H}$  and the complete intersection of two general members of  $\mathcal{H}$ . Since the singular locus of  $V$  has at most dimension  $\delta$ , [Puk02, Proposition 5] implies that for every close subvariety  $T \subset V$  we have  $e_T(D) \leq r$  if  $\dim T \leq \delta + 1$ , and  $e_T(B) \leq r^2$  if  $\dim T \leq \delta + 2$  ( $e_T(S)$  denoting the *multiplicity* of a scheme  $S$  along a subvariety  $T$ ). The first of these inequalities implies that the pair  $(V, \frac{1}{r}B)$  has terminal singularities away from a set of dimension  $\delta$  (e.g., see [dF12, Proposition 8.8]). On the other hand, the Noether–Fano Inequality implies that the pair  $(V, B)$  has canonical threshold

$$c := \text{can}(V, B) < 1/r.$$

Moreover, if  $E_1$  is a (prime) divisor over  $X$  computing the canonical thresholds, that is, with log-discrepancy

$$a_{E_1}(V, cB) = 1$$

over  $(X, cB)$ , then the center of  $E_1$  in  $V$  has dimension at most  $\delta + 1$  because of the bound on the multiplicities of  $D$ .

Fix a point  $P$  in the center of  $E_1$  in  $V$ , let

$$\mathbb{P}^{N-\delta-1} \subset \mathbb{P}^N$$

be a general linear subspace of dimension  $N - \delta - 1$  passing through  $P$ , and let  $Y \subset \mathbb{P}^{N-\delta-1}$  be the restriction of  $V$  to this subspace. Then the pair  $(Y, cB_Y)$  is terminal away from finitely many points, including  $P$  where the pair is still not canonical by Inversion of Adjunction (or, equivalently, by the Connectedness Theorem). It follows that there is a divisor  $E_2$  over  $Y$ , with center  $P$ , such that

$$a_{E_2}(Y, cB|_Y) \leq 1.$$

We then take one more hyperplane section: let

$$\mathbb{P}^n = \mathbb{P}^{N-\delta-2} \subset \mathbb{P}^{N-\delta-1}$$

be a general hyperplane through  $P$  (for simplicity, we have set  $n := N - \delta - 2$ ). Let  $X \subset \mathbb{P}^n$  be the restriction of  $Y$ . Applying again Inversion of Adjunction (or by the Connectedness Theorem), we find now a divisor  $E_3$  over  $X$ , with center  $P$ , such that

$$a_{E_3}(X, cB|_X) \leq 0.$$

Furthermore,  $e_T(B|_X) \leq r^2$  for all positive dimensional subvarieties  $T \subset X$  (cf. [dF12, Propositions 8.7 and 8.5(ii)]).

The above computations were done assuming that  $V$  was terminal but not birationally superrigid. Suppose then, for a moment, that  $V$  is not terminal. This means that there is a divisor  $E_1$  over  $V$  such that  $a_{E_1}(V, \emptyset) \leq 1$ , whose center is contained in the singular locus of  $V$ . Then the same proof given assuming that  $V$  is terminal but not birationally superrigid, simplified in fact by the fact that we can replace  $B$  with the empty set, goes through to produce a contradiction, thus proving that  $V$  must have terminal singularities. (Equivalently, one can follow verbatim the same proof by just taking  $B$ , in the non-terminal case, to be a codimension two subscheme of  $V$  cut out by two general equations of the same degree  $r$  vanishing on the center of  $E_1$ .) Thus, we can assume henceforth that  $V$  has terminal singularities.



## 4. CORE OF THE PROOF

**4.1. Background: Arc spaces and maximal divisorial sets.** Standard references are [ELM04, EM09]; the reader may also refer to [dF12, Sections 3 and 4] for a short, comprehensive introduction (in the smooth case) to the basic tools needed here.

Briefly, to each variety  $X$  we associate its jet schemes  $J_m X$  and arc space  $J_\infty X$ , which respectively parametrize  $m$ -jets  $\text{Spec } \mathbb{C}[t]/(t^{m+1}) \rightarrow X$  and formal arcs  $\text{Spec } \mathbb{C}[[t]] \rightarrow X$ . For any set  $W \subset J_\infty X$ , we denote by  $W_m$  the projection of  $W$  via the truncation map  $J_\infty X \rightarrow J_m X$ . A set  $W \subset J_\infty X$  is a cylinder if it is constructible and is equal to the inverse image of its image  $W_m \subset J_m X$  for some  $m$ ; the set is said to be a quasi-cylinder if equality holds away from the set of arcs fully contained in the singular locus of  $X$ .

Given a smooth prime divisor  $E$  on a resolution  $f: \tilde{X} \rightarrow X$  of  $X$ , and a positive integer  $q$ , we consider the maximal divisorial set

$$W^q(E) := \overline{f_\infty(\text{Cont}^q(E))} \subset J_\infty X.$$

Here  $\text{Cont}^q(E)$  is the locus of arcs on  $\tilde{X}$  with contact order  $q$  along  $E$ ,  $f_\infty: J_\infty \tilde{X} \rightarrow J_\infty X$  is the map induced on arc spaces by  $f$ , and the closure in the right-hand side is taken with respect to the Zariski topology of  $J_\infty X$ . The set  $W^q(E)$  only depends on the valuation  $q \text{val}_E$ , and is a quasi-cylinder in  $J_\infty X$ .

The valuation  $q \text{val}_E$  can be reproduced from the set  $W^q(E)$  by considering the order of contact with the generic point of  $W^q(E)$ , which is a  $K$ -valued arc  $\text{Spec } K[[t]] \rightarrow X$  for some field extension  $\mathbb{C}(X) \subset K$ . More generally, to each closed quasi-cylinder  $C \subset J_\infty X$  that is irreducible and is not contained in the arc space of the singular locus, one can associate a valuation  $\text{val}_C$  on  $X$ . This valuation is divisorial, which means that  $\text{val}_C = q \text{val}_E$  for some divisor  $E$  over  $X$  and some positive integer  $q$ , and we have an inclusion  $C \subset W^q(E)$  (which justifies the terminology of maximal divisorial set). For the complete correspondence between quasi-cylinders, maximal divisorial sets, and divisorial valuations, we refer to [ELM04, dFEI08]. If  $X$  is smooth, then the discrepancy along  $E$  is also encoded in its maximal divisorial sets, as we have

$$\text{codim}(W^q(E), J_\infty X) = q(k_E(X) + 1)$$

where the codimension is here intended topologically, and thus maximal divisorial sets carry enough information to recover the log discrepancy of a pair  $(X, Z)$  along the corresponding exceptional divisor. A similar formula holds also for singular varieties.

The first jet space  $J_1 X$  is naturally identified with the tangent cone bundle over  $X$ , meaning that the fiber of  $J_1 X$  over a point  $P \in X$  is identified with the tangent cone  $C_P X$  of  $X$  at  $P$ . In general, for every  $m \geq 1$ , we define the  $m$ -th tangent cone bundle  $C^{(m)} X$  of  $X$  to be the inverse image in  $J_m X$  of the trivial section of  $J_{m-1} X \rightarrow X$ , so that the fiber  $C_P^{(m)} X$  of  $C^{(m)} X$  over  $P$  is the same as the fiber of  $J_m X$  over the constant  $(m-1)$ -th jet of  $X$  at  $P$ . It is immediate to check that  $C^{(m)} X$  parametrizes maps  $\text{Spec } \mathbb{C}[t^m]/(t^{m+1}) \rightarrow X$ , and that for every  $m, q \geq 1$ , the isomorphisms of rings  $A[t^m]/(t^{m+1}) \rightarrow A[t^p]/(t^{p+1})$  sending  $a + bt^m$  to  $a + bt^p$  for every  $\mathbb{C}$ -algebra  $A$  give rise to a fiber-wise linear isomorphism

$$\psi_{m,p}^X: C^{(m)} X \rightarrow C^{(p)} X.$$

These definitions generalize those of  $m$ -th tangent bundle/space given in [dF12] when  $X$  is smooth, where the notation  $T^{(m)} X$  and  $T_P^{(m)} X$  is used instead.

Suppose that  $E$  is a prime divisor over  $X$  with center equal to a point  $P \in X$ . Let  $\Gamma_E$  be the center of  $E$  on the blow-up  $\text{Bl}_P X$ . Since  $\Gamma_E$  is contained in the exceptional divisor, we



can then take its cone  $\widehat{\Gamma}_E$  inside the tangent cone  $C_P X$ . Let  $W := W^1(E) \subset J_\infty X$ , and let

$$\mu := \min\{m \mid \dim W_m \geq 2\}.$$

Note that  $\mu = \text{val}_E(P)$  and  $W_\mu \subset C_P^{(\mu)} X$ . Extending the properties studied in [dF12, Section 4] to the singular case, we see that the set

$$\psi_{\mu,1}^X(W_\mu) \subset C_P X$$

is a dense, constructible subcone of  $\widehat{\Gamma}_E$ . The non-zero elements in  $\psi_{\mu,1}^X(W_\mu) \subset \widehat{\Gamma}_E$  are said to be the principal tangent vectors of  $E$  at  $P$ . The elements in  $\Gamma_E$  given by homogeneous classes of non-zero elements in  $\psi_{\mu,1}^X(W_\mu)$  are the principal tangent directions of  $E$  at  $P$ . This generalizes [dF12, Definition 3.7] to the singular case.

**4.2. Step 2 of the proof: First projection and degeneration.** Back to the proof of Theorem 1.2, and to the setting of Section 3.2, consider the maximal divisorial set

$$W := W^1(E_3) \subset J_\infty X.$$

Since  $P$  is an isolated singularity, every quasi-cylinder of  $J_\infty X$  is a cylinder. In particular, we can fix a sufficiently large integer  $m$  so that  $W$  is a cylinder over  $W_m \subset J_m X$ .

Fix affine coordinates  $(x_1, \dots, x_n)$  on  $\mathbb{A}^n \subset \mathbb{P}^n$  centered at  $P$ , and consider the  $\mathbb{C}^*$ -action  $x_i \mapsto sx_i$ , where  $s \in \mathbb{C}^*$ . The action lifts to  $J_m \mathbb{A}^n$ . Regarding  $W_m$  as a close subset of  $J_m \mathbb{A}^n$ , let  $(W_m)^0 \subset J_m \mathbb{A}^n$  be its (set-theoretic) limit obtained by taking a flat degeneration as  $s \rightarrow 0$ . Let

$$\mu := \min\{q \mid \dim((W_m)^0)_q \geq 2\}.$$

Note that  $((W_m)^0)_\mu \subset T_P^{(\mu)} \mathbb{A}^n$ . We pick a non-zero tangent vector

$$\xi \in \psi_{\mu,1}^{\mathbb{A}^n}(((W_m)^0)_\mu) \subset T_P \mathbb{A}^n.$$

We take a general linear projection

$$\sigma: \mathbb{A}^n \rightarrow \mathbb{A}^{n-1},$$

which we regard as a rational map  $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ , and consider the induced finite map

$$g: X \rightarrow \mathbb{P}^{n-1}.$$

Let  $P' := g(P) \in \mathbb{P}^{n-1}$ . Via the inclusion of function fields  $\mathbb{C}(\mathbb{P}^{n-1}) \subset \mathbb{C}(X)$ , the valuation  $\text{val}_{E_3}$  restricts to a divisorial valuation on  $\mathbb{P}^{n-1}$  which can be written as  $q_4 \text{val}_{E_4}$  for some divisor  $E_4$  with center  $P'$  and some positive integer  $q_4$ . We fix resolutions  $\widetilde{X} \rightarrow X$  and  $\widetilde{\mathbb{P}}^{n-1}$  fitting in a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f} & X \\ \widetilde{g} \downarrow & & \downarrow g \\ \widetilde{\mathbb{P}}^{n-1} & \longrightarrow & \mathbb{P}^{n-1}. \end{array}$$

We can assume without loss of generality that  $E_3$  and  $E_4$  appear, respectively, as divisors on the models  $\widetilde{X}$  and  $\widetilde{\mathbb{P}}^{n-1}$ .

Let

$$W' := W^{q_4}(E_4) \subset J_\infty \mathbb{P}^{n-1}.$$

Note that  $W'$  is contained in the fiber over  $P'$ , which is contained in  $J_\infty \mathbb{A}^{n-1}$ . After a linear change of coordinates, we can assume that  $(x_1, \dots, x_n)$  are chosen so that  $(x_1, \dots, x_{n-1})$  come from coordinates on  $\mathbb{A}^n$ , centered at  $P'$ . Let  $(W')^0 \subset J_\infty \mathbb{A}^n$  be the (set-theoretic)

homogeneous limit of  $W'$  under the  $\mathbb{C}^*$ -action  $x_i \mapsto sx_i$  when  $s \rightarrow 0$ , as defined in [dF12, Section 5].

**Lemma 4.1.**  $\sigma_\mu(((W_m)^0)_\mu) = ((W')^0)_\mu$ .

*Proof.* The fact that  $\text{val}_{E_3}$  restricts to  $q_4 \text{val}_{E_4}$  means that  $\tilde{g}$  has ramification order  $q_4$  at the generic point of  $E_3$ . Then, after restricting over some open neighborhood  $U \subset \tilde{\mathbb{P}}^{n-1}$  of the generic point of  $E_4$ , every arc on  $\tilde{g}^{-1}(U)$  with contact one along  $E_3$  is mapped to an arc on  $U$  with contact  $q_4$  along  $E_4$ , and conversely, any such arc on  $U$  is realized in this way. This implies that  $\tilde{g}_\infty(\text{Cont}^1(E_3)) = \text{Cont}^{q_4}(E_4)$ , and therefore  $g_\infty(W) = W'$ . It also follows that  $W'$  is a cylinder over  $(W')_m$ , and thus we have  $((W')^0)_m = (W'_m)^0$  by [dF12, Lemma 5.3].

Since  $\sigma$  is a general projection, we can assume that the tangent cone  $C_P X \subset T_P \mathbb{A}^n$  does not contain the kernel of the linear map  $T_P \mathbb{A}^n \rightarrow T_{P'} \mathbb{A}^{n-1}$ . This means that if  $\overline{C_P X}$  is the projective closure of  $C_P X$  in  $\text{Proj } \mathbb{C}[u, x'_1, \dots, x'_n]$  where  $(x'_1, \dots, x'_n)$  are the coordinates induced by  $(x_1, \dots, x_n)$  on  $T_P \mathbb{A}^n$ , then the point of homogeneous coordinates  $(0 : \dots : 0 : 1)$  is not in  $\overline{C_P X}$ . Equivalently, using the coordinates  $(x_1^{(q)}, \dots, x_n^{(q)})$  induced on  $T_P^{(q)} \mathbb{A}^n$ , the point  $(0 : \dots : 0 : 1) \in \text{Proj } \mathbb{C}[u, x_1^{(q)}, \dots, x_n^{(q)}]$  is not contained in the projective closure  $\overline{C_P^{(q)} X}$  of  $C_P^{(q)} X$  for any  $q \geq 1$ . This point being settled, the proof follows the same arguments of the proof of [dF12, Lemma 9.1] by simply replacing  $\overline{C_P^{(q)} X}$  with  $\overline{C_P^{(q)} X}$ .  $\square$

By construction,  $((W')^0)_\mu \subset T_{P'}^{(m)} \mathbb{A}^{n-1}$ . By taking the projection generally, we can ensure that  $\xi$  maps to a non-zero vector  $\xi' \in T_{P'} \mathbb{A}^{n-1}$ . Then, by Lemma 4.1,

$$\xi' \in \psi_{\mu,1}^{\mathbb{A}^{n-1}}(((W')^0)_\mu).$$

Let  $C$  be an irreducible component of  $(W')^0$  such that  $\xi' \in \psi_{\mu,1}^{\mathbb{A}^{n-1}}(C_\mu)$ . Since  $C$  is a cylinder in  $J_\infty \mathbb{P}^{n-1}$ , there is a divisor  $E_5$  over  $\mathbb{P}^{n-1}$  with center  $P'$ , and a positive integer  $q_5$ , such that  $\text{val}_C = q_5 \text{val}_{E_5}$ .

The projection  $X \rightarrow \mathbb{P}^{n-1}$  is the step that allows us to get rid of the singularities of  $X$ . The (negative) contribution coming from the singularity, measure in terms of the Jacobian, is encoded in the following formula on log discrepancies.

**Lemma 4.2.**  $q_4(k_{E_4}(\mathbb{P}^{n-1}) + 1) = k_{E_3}(X) + 1 + \text{val}_{E_3}(\text{Jac}_X)$ .

*Proof.* By evaluating the identity

$$K_{\tilde{X}/\tilde{\mathbb{P}}^{n-1}} + \tilde{g}^* K_{\tilde{\mathbb{P}}^{n-1}/\mathbb{P}^{n-1}} = K_{\tilde{X}/X} + f^* K_{X/\mathbb{P}^{n-1}}$$

at the generic point of  $E$  we get

$$q_4 k_{E_4}(\mathbb{P}^{n-1}) + q_1 - 1 = k_{E_3}(X) + \text{val}_{E_3}(K_{X/\mathbb{P}^{n-1}}).$$

The lemma then follows from the fact that if  $X$  is defined in  $\mathbb{A}^n$  by  $h(x_1, \dots, x_n) = 0$ , then the ramification divisor  $K_{X/\mathbb{P}^{n-1}}$  is defined on  $X \cap \mathbb{A}^n$  by the vanishing of  $\partial h / \partial x_n$ , which is a linear combination of a set of generators of the Jacobian ideal. Taking a general projection  $\mathbb{A}^n \rightarrow \mathbb{A}^{n-1}$  correspond to this element being a general linear combination, and hence we have

$$\text{val}_{E_3}(K_{X/\mathbb{P}^{n-1}}) = \text{val}_{E_3}(\text{Jac}_X)$$

for a general projection  $X \rightarrow \mathbb{P}^{n-1}$ .  $\square$

We are now ready to come back to our bounds on log discrepancies. For short, denote  $Z := B|_X$ . We established at the end of the previous section that  $a_{E_3}(X, cZ) \leq 0$  where  $E_3$  is a divisor with center  $P$ . By our hypothesis on the singularities of  $V$ , there is an inclusion

$$(\mathfrak{m}_{X,P})^{\nu-1} \subset \overline{(\text{Jac}_X \cdot \mathcal{O}_{X,P})}$$

This implies that  $\text{val}_{E_3}(\text{Jac}_X) \leq (\nu - 1) \text{val}_E(\mathfrak{m}_{X,P})$ , and hence, denoting by  $J_X \subset X$  the scheme defined by  $\text{Jac}_X$ , we have

$$a_{E_3}(X, cZ + (\nu - 1)P - J_X) \leq 0.$$

Consider the scheme-theoretic image  $Z' := g(Z) \subset \mathbb{P}^{n-1}$ . Then  $q_4 \text{val}_{E_4}(Z') = \text{val}_{E_3}(g^{-1}(Z')) \geq \text{val}_{E_3}(Z)$ . Similarly, we have  $q_4 \text{val}_{E_4}(P') = \text{val}_{E_3}(g^{-1}(P')) \geq \text{val}_{E_3}(P)$ . Combining these inequalities with the previous lemma, we then obtain

$$a_{E_4}(\mathbb{P}^{n-1}, cZ' + (\nu - 1)P') \leq 0.$$

Finally, if  $(Z')^0 \subset \mathbb{P}^{n-1}$  is the flat limit of  $Z'$  obtained by taking a degeneration under the  $\mathbb{C}^*$ -action  $x_i \mapsto sx_i$  as  $s \rightarrow 0$ , then [dF12, Proposition 5.5] implies that

$$a_{E_5}(\mathbb{P}^{n-1}, c(Z')^0 + (\nu - 1)P') \leq 0.$$

## 5. CONCLUSION OF THE PROOF

**5.1. Background: Multiplier ideals.** A standard reference is [Laz04]. Given an effective pair  $(X, D)$  where  $X$  is a normal variety with  $\mathbb{Q}$ -Cartier canonical class, and  $D$  is an effective  $\mathbb{R}$ -divisor, we define the multiplier ideal of the pair to be the ideal sheaf

$$\mathcal{J}(X, D) := f_* \mathcal{O}_{\tilde{X}}(\lceil K_{\tilde{X}/X} - f^*D \rceil).$$

Here  $f: \tilde{X} \rightarrow X$  is a log resolution of the pair, so that the exceptional locus has pure codimension one and the union of the supports of  $K_{\tilde{X}/X}$  and  $f^*D$  has simple normal crossings, and the round-up in the right-hand side is taken component-wise (the definition is independent of the choice of resolution).

A proof of the following vanishing theorem, generally attributed to Nadel, can be found for instance in [Laz04].

**Theorem 5.1** (Nadel Vanishing Theorem). *With the above notation, suppose that  $L$  is a Cartier divisor such that  $L - D$  is nef and big. Then*

$$H^i(X, \mathcal{O}_X(L) \otimes \mathcal{J}(X, D)) = 0 \quad \text{for all } i > 0.$$

The next result gives a rare instance where Inversion of Adjunction holds for non-effective pairs.

**Theorem 5.2** (Special Inversion of Adjunction [dF12]). *Let  $X = \mathbb{A}^n$ , with affine coordinates  $(x_1, \dots, x_n)$ . Let  $E$  be a prime divisor over  $X$  with center at the origin  $O = (0, \dots, 0)$ , and assume that  $\text{val}_E$  is invariant under the homogeneous  $\mathbb{C}^*$ -action  $x_i \mapsto sx_i$ . Let  $Y = \mathbb{A}^{n-e} \subset X$  be a linear subspace of codimension  $e < n$  that is tangent to a principal tangent directions of  $E$  at  $O$ , and let  $D$  is an effective  $\mathbb{R}$ -divisor on  $X$  not containing  $Y$  in its support. Then there is a divisor  $F$  over  $Y$  with center  $O$ , and a positive integer  $q$ , such that*

$$qa_F(Y, D|_Y - eP) \leq a_E(X, D).$$

In particular, if  $a_E(X, D) \geq 0$ , then  $a_F(Y, D|_Y - eP) \leq a_E(X, D)$ .

*This theorem can be interpreted as a special restriction property for multiplier ideals. Indeed, if we assume that, in the hypothesis of the theorem,  $a_E(X, D) \leq a$  for some  $a \in \{0, 1, \dots, e\}$ , then the theorem implies that  $(\mathfrak{m}_{Y,O})^{e-a} \notin \mathcal{J}(Y, D|_Y)$ . See [dF12, Section 6] for further discussion of this property.*

**5.2. Step 3 of the proof: Second projection and last restriction.** This part of the proof is essentially the same as the last part of the proof of [dF12, Theorem 7.4]. We outline it for the convenience of the reader.

Returning to where we left in Section 4.2, take a general linear projection

$$\tau: \mathbb{A}^{n-1} \rightarrow \mathbb{A}^{n-2}$$

which, again, we regard as a rational map  $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-2}$ . The push-forward  $A^0 := \tau_*[(Z')^0]$  is a divisor of degree  $Nr^2$  on  $\mathbb{P}^{n-2}$ , and is a cone with vertex  $Q := \tau(P')$ .

By [dF12, Proposition 5.2],  $(Z')^0$  is the image of the homogeneous limit  $Z^0$  of  $Z$  obtained under the action  $x_i \mapsto sx_i$  as  $s \rightarrow 0$ . Since  $Z^0$  is complete intersection in  $\mathbb{P}^n$ , [dF12, Lemma 9.2] applies to our setting, and hence  $(Z')^0$  is a Cohen–Macaulay scheme. We can then apply [dFEM03, Theorem 1.1] (in the more precise form stated in [dF12, Theorem 8.1]), which implies that the valuation  $\text{val}_{E_5}$  restrict to a  $\mathbb{C}^*$ -invariant divisorial valuation  $q_6 \text{val}_{E_6}$  on  $\mathbb{P}^{n-2}$  and, bearing in mind that  $q_6 \text{val}_{E_6}(Q) = \text{val}_{E_5}(\tau^{-1}(Q)) \geq \text{val}_{E_5}(P)$ , we have

$$a_{E_6}(\mathbb{P}^{n-2}, \frac{c^2}{2}A^0 + (\nu - 1)Q) \leq 0.$$

Note that the tangent direction determined by the image  $\xi'' \in T_Q \mathbb{P}^{n-2}$  of  $\xi'$  is contained in the center of  $E_6$  on the blow-up  $\text{Bl}_Q \mathbb{P}^{n-2}$ . Let  $L'' \subset \mathbb{P}^{n-2}$  be the line with tangent vector  $\xi''$  at  $Q$ . Arguing as in [dF12, Lemma 8.3], we obtain  $e_{L''}(A^0) \leq r^2$ . Recall that the set of principal tangent directions of  $E_6$  is dense in the center of  $E_6$  on the blow-up. We can therefore pick a principal tangent vector  $\xi^*$  of  $E_6$  at  $Q$  that is close enough to  $\xi''$  so that, if  $L^* \subset \mathbb{P}^{n-2}$  is the line with tangent vector  $\xi^*$  at  $Q$ , then  $e_{L^*}(A^0) \leq r^2$ . Let then  $\mathbb{P}^2 \subset \mathbb{P}^{n-2}$  be a general plane containing  $L^*$ . By [dF12, Proposition 8.5], we have  $e_T(A^0|_{\mathbb{P}^2}) \leq r^2$  for every irreducible curve  $T \in \mathbb{P}^2$ , and hence the multiplier ideal

$$\mathcal{J}(\mathbb{P}^2, \frac{c^2}{2}A^0|_{\mathbb{P}^2})$$

vanishes on a zero dimensional scheme.

By Theorem 5.2, there is a divisor  $E_7$  over  $\mathbb{P}^2$ , with center  $Q$ , such that

$$a_{E_7}(\mathbb{P}^2, \frac{c^2}{2}A^0|_{\mathbb{P}^2} - (n - \nu - 3)Q) \leq 0.$$

This implies that if  $\mathfrak{q}$  is the primary component of  $\mathcal{J}(\mathbb{P}^2, \frac{c^2}{2}A^0|_{\mathbb{P}^2})$  that is co-supported at  $Q$ , then

$$(\mathfrak{m}_{\mathbb{P}^2, Q})^{n-\nu-3} \not\subset \mathfrak{q}.$$

Since  $\mathfrak{q}$  is a homogeneous ideal in the affine coordinates centered at  $Q$ , this means that among the homogeneous generators of the  $\mathbb{C}$ -vector space  $\mathcal{O}_{\mathbb{P}^2}/\mathfrak{q}$  there must be some elements of degree at least  $n - \nu - 5$ . On the other hand, the Nadel Vanishing Theorem implies that there is a surjection

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k - 3)) \twoheadrightarrow H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}/\mathfrak{q})$$

for every integer  $k > Nr^2c^2/2$ . Putting together, we get  $Nr^2c^2/2 \geq n - \nu - 1$ . Recalling that  $c < 1/r$  and  $n = N - \delta - 2$ , this gives  $\delta + \nu > \frac{1}{2}N - 3$ . This however is in contradiction with our hypothesis on the singularities of  $X$ .

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